Monte Carlo Methods for Estimating Economic Capital

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Abstract

Economic capital (EC) is a risk measure used by financial firms to help determine capital levels to protect (with high probability) against large unexpected losses of credit portfolios. Also known as the credit, relative, or mean-adjusted value-at-risk (VaR), the EC is defined as the difference between an extreme quantile (e.g., with quantile level p = 0.999 or .9998) and the mean of the distribution of the portfolio's loss over a given time horizon (e.g., one year). (Recall that for p in (0,1), the p-quantile, or VaR, is the inverse of the distribution function evaluated at p; e.g., the median is the 0.5-quantile.)
We consider estimating EC via Monte Carlo methods. Simple random sampling (SRS) does well to estimate the mean, but it performs poorly for the extreme quantile, leading to the SRS EC estimator having large variance. In contrast, importance sampling (IS) can be inefficient because an IS measure designed to estimate an extreme quantile can produce mean estimators with greatly inflated variance. This motivates applying methods that combine SRS and IS.

One such approach is measure-specific importance sampling (MSIS), which independently estimates the extreme quantile and the mean using IS and SRS, respectively, with their difference yielding the MSIS EC estimator. Another method, importance sampling with a defensive mixture (ISDM), applies IS with a change of measure corresponding to a mixture of a new distribution and the original. We further consider a double estimator (DE), which estimates both the extreme quantile and the mean with both SRS and IS, and the DE estimator takes a linear combination of the four estimators.

We provide theoretical comparisons of the EC estimators when the loss is a sum of m independent and identically distributed (i.i.d.) light-tailed random variables, where the IS in any method applies exponential twisting. We establish asymptotic properties of our EC estimators as m tends to infinity with the quantile level p equivalent to p_m = 1 - exp(-b m) simultaneously approaching 1 for any positive constant b. In addition to its theoretical convenience, this asymptotic regime also has practical relevance: credit portfolios are commonly exposed to thousands of obligors,and banks often use p larger than 0.999 in their EC computations. We analyze the asymptotic behavior of the relative errors (REs) of the different EC estimators, where the RE is the square root of an estimator's asymptotic variance from its central limit theorem, normalized by dividing by the estimand. We prove that for MSIS and ISDM, the EC estimators have asymptotic variances and REs that behave polynomially in m, with MSIS leading to vanishing RE and ISDM yielding bounded RE. In

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contrast,SRS, IS, and DE produce EC estimators having asymptotic variances and REs that grow exponentially in m, with DE behaving asymptotically as the worst of its constituent SRS and IS estimators. Our theoretical analysis for the i.i.d. sum model agrees with numerical experiments on a more complicated problem, a 10-factor model of a credit portfolio having 1000 dependent obligors, with dependence induced through a Gaussian copula.

Our theoretical study of the i.i.d. sum model further separately examines the asymptotic behavior of the various estimators of the mean and the extreme quantile. In estimating the mean, the IS estimator (via an exponential twist with any fixed non-zero twisting parameter) has variance and RE that increase exponentially in m, but the ISDM estimator's variance behaves polynomially and its RE remains bounded as m tends to infinity.

This analysis also provides insights into so-called "green simulation" methods. The goal here is to estimate the means of various models that differ in only their input distributions. Green simulation reuses outputs from previous experiments by weighting
each with a likelihood ratio (LR) to obtain an unbiased estimator. Our theory indicates that when an output is generated from a large number of independent random inputs, green simulation can produce estimators with large variance if the LR is simply the ratio of the densities under the different input distributions, as in IS. But in contrast, ISDM can be quite effective in controlling the variance.
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