Limit Theorems for Cloning Algorithms

Stefan Grosskinsky

TU Delft - DIAM

May 21, 2021

RESIM 2021

The Team

Letizia Angeli

Adam Johansen

Andrea Pizzoferrato

- L. Angeli, S. Grosskinsky, A.M. Johansen. Limit theorems for cloning algorithms. Stoch. Proc. Appl. **138**: 117-152 (2021)
- L. Angeli, S. Grosskinsky, A.M. Johansen, A. Pizzoferrato. Rare event simulation for stochastic dynamics in continuous time. Journal of Statistical Physics **176**(5): 1185–1210 (2019)

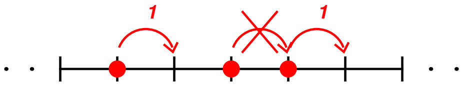
Outline

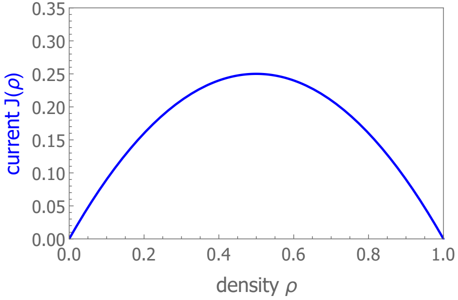
- Motivation from statistical physics
- SMC for continuous-time jump processes (Feynman-Kac models)
- Cloning algorithm, comparison to other particle approximations
- Current fluctuations for IPS

Motivation from Statistical Physics

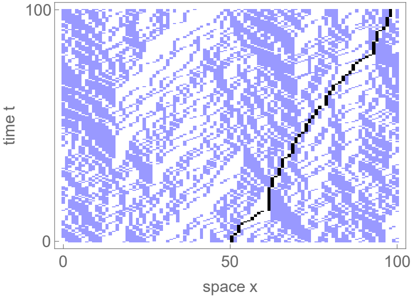
Current large deviations in lattice gases/dynamic phase transitions

TASEP





$J(\rho)$ typical current

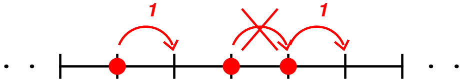


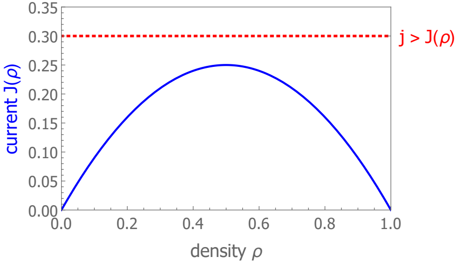
typical particle trajectories

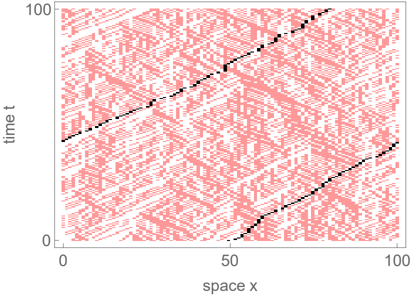
Motivation from Statistical Physics

Current large deviations in lattice gases/dynamic phase transitions

TASEP





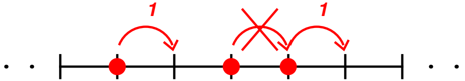


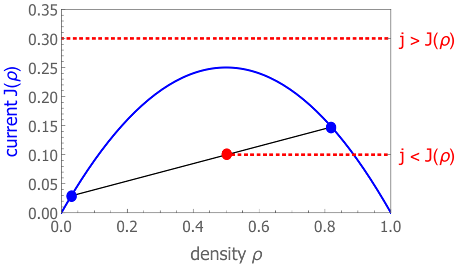
$J(\rho)$ typical current , $j > J(\rho)$ atypical current **correlated trajectories**

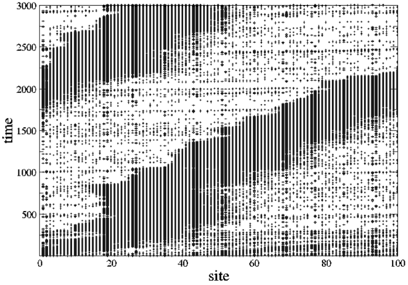
Motivation from Statistical Physics

Current large deviations in lattice gases/dynamic phase transitions

TASEP



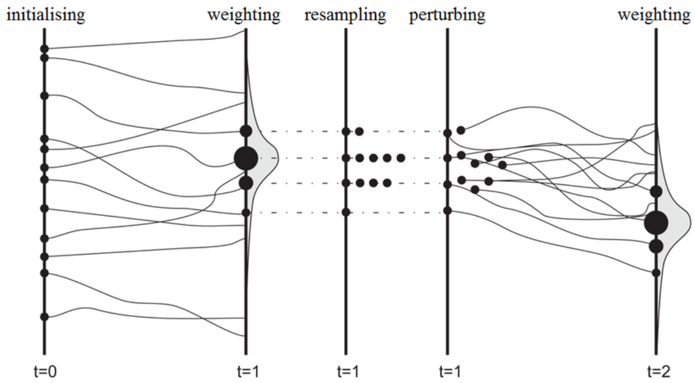




[Giardina et al. (2006)]

$J(\rho)$ typical current , $j < J(\rho)$ atypical current **phase separation**

Sequential Monte Carlo methods



[D. Alvares, PhD thesis, researchgate.net]

Particle filters

[Crisan, Lyons (1997); Del Moral, Miclo (2000); Del Moral, Doucet, Jasra (2006); Rousset (2006); . . .]

dynamic rare events/cloning algorithms

[Giardina, Kurchan, Peliti (2006); Lecomte, Tailleur (2007); Pérez-Espirages, Hurtado (2019); . . .]

Mathematical setting

Continuous-time Markov jump process $(X_t : t \geq 0)$, $\mathbb{P}_x, \mathbb{E}_x$

state space E, locally compact Polish space (e.g. $\mathbb{R}^d$, $\mathbb{N}_0^L$ or $\{0,1\}^{\mathbb{Z}}$)

transition kernel $W(x, dy)$, $w(x) = \int_E W(x, dy) \leq \bar{w} < \infty$

generator $\mathcal{L}f(x) = \int_E W(x, dy)[f(y) - f(x)], \quad f \in C_b(E), x \in E$

Mathematical setting

Continuous-time Markov jump process $(X_t : t \geq 0)$, $\mathbb{P}_x, \mathbb{E}_x$

state space E, locally compact Polish space (e.g. $\mathbb{R}^d$, $\mathbb{N}_0^L$ or $\{0,1\}^{\mathbb{Z}}$)

transition kernel $W(x, dy)$, $w(x) = \int_E W(x, dy) \leq \bar{w} < \infty$

generator $\mathcal{L}f(x) = \int_E W(x, dy)[f(y) - f(x)], \quad f \in C_b(E), x \in E$

distribution

$$\mu_t(f) := \mathbb{E}_{\mu_0}\big[f(X_t)\big] \quad \text{with} \quad \frac{d}{dt}\mu_t(f) = \mu_t(\mathcal{L}f) \ , \quad f \in C_b(E)$$

Mathematical setting

Continuous-time Markov jump process $(X_t : t \geq 0)$, $\mathbb{P}_x, \mathbb{E}_x$

state space E, locally compact Polish space (e.g. $\mathbb{R}^d$, $\mathbb{N}_0^L$ or $\{0,1\}^{\mathbb{Z}}$)

transition kernel $W(x, dy)$, $w(x) = \int_E W(x, dy) \leq \bar{w} < \infty$

generator $\mathcal{L}f(x) = \int_E W(x, dy)[f(y) - f(x)], \quad f \in C_b(E), x \in E$

distribution

$$\mu_t(f) := \mathbb{E}_{\mu_0}\big[f(X_t)\big] \quad \text{with} \quad \frac{d}{dt}\mu_t(f) = \mu_t(\mathcal{L}f) \ , \quad f \in C_b(E)$$

Assumption. Asymptotic stability

There exist $C > 0$, $\alpha \in (0,1)$, such that for all $\mu_0 \in \mathcal{P}(E)$

$$\big\|\mu_t(f) - \mu_\infty(f)\big\| \leq C\|f\|\alpha^t \ , \quad t \geq 0 \ ,$$

where $\mu_\infty \in \mathcal{P}(E)$ is the unique **stationary distribution**.

Feynman-Kac model

potential $\mathcal{V} \in C_b(E)$, $\mathcal{L}^{\mathcal{V}} f(x) := \mathcal{L} f(x) + \mathcal{V}(x) f(x)$

Feynman-Kac measures $(\nu_t : t \geq 0)$ where $\nu_0 = \mu_0$

$$\nu_t(f) := \mathbb{E}_{\mu_0}\Big[f(X_t)\, e^{\int_0^t \mathcal{V}(X_s)ds}\Big] \quad \text{with} \quad \frac{d}{dt}\nu_t(f) = \nu_t(\mathcal{L}^{\mathcal{V}} f)$$

non-conservative $\mathcal{L}^{\mathcal{V}} 1(x) = 0 + \mathcal{V}(x) \neq 0$

Feynman-Kac model

potential $\mathcal{V} \in C_b(E)$, $\mathcal{L}^{\mathcal{V}} f(x) := \mathcal{L} f(x) + \mathcal{V}(x) f(x)$

Feynman-Kac measures $(\nu_t : t \geq 0)$ where $\nu_0 = \mu_0$

$$\nu_t(f) := \mathbb{E}_{\mu_0}\Big[f(X_t)\, e^{\int_0^t \mathcal{V}(X_s)ds} \Big] \quad \text{with} \quad \frac{d}{dt}\nu_t(f) = \nu_t(\mathcal{L}^{\mathcal{V}} f)$$

non-conservative $\mathcal{L}^{\mathcal{V}} 1(x) = 0 + \mathcal{V}(x) \neq 0$

principal eigenvalue $\lambda \in \mathbb{R}$ with left eigenvector $\mu_\infty^{\mathcal{V}} \in \mathcal{P}(E)$

$$\lim_{t\to\infty} \frac{1}{t} \log \nu_t(1) = \lambda = \mu_\infty^{\mathcal{V}}(\mathcal{L}^{\mathcal{V}} 1) = \mu_\infty^{\mathcal{V}}(\mathcal{V})$$

Feynman-Kac model

potential $\mathcal{V} \in C_b(E)$, $\mathcal{L}^{\mathcal{V}} f(x) := \mathcal{L} f(x) + \mathcal{V}(x) f(x)$

Feynman-Kac measures $(\nu_t : t \geq 0)$ where $\nu_0 = \mu_0$

$$\nu_t(f) := \mathbb{E}_{\mu_0}\Big[f(X_t)\, e^{\int_0^t \mathcal{V}(X_s)ds} \Big] \quad \text{with} \quad \frac{d}{dt}\nu_t(f) = \nu_t(\mathcal{L}^{\mathcal{V}} f)$$

non-conservative $\mathcal{L}^{\mathcal{V}} 1(x) = 0 + \mathcal{V}(x) \neq 0$

principal eigenvalue $\lambda \in \mathbb{R}$ with left eigenvector $\mu_\infty^{\mathcal{V}} \in \mathcal{P}(E)$

$$\lim_{t\to\infty} \frac{1}{t} \log \nu_t(1) = \lambda = \mu_\infty^{\mathcal{V}}(\mathcal{L}^{\mathcal{V}} 1) = \mu_\infty^{\mathcal{V}}(\mathcal{V})$$

normalized measures $\mu_t^{\mathcal{V}}(f) = \mu_t(f) := \nu_t(f)/\nu_t(1)$

$$\frac{d}{dt}\mu_t(f) = \mu_t\Big(\mathcal{L} f + \mathcal{V} f - \mu_t(\mathcal{V}) f \Big) \qquad (\star)$$

asymptotic stability $\big\| \mu_t(f) - \mu_\infty^{\mathcal{V}}(f) \big\| \leq C \|f\| \alpha^t$, $t \geq 0$, $\alpha \in (0,1)$

McKean interpretation

$$\frac{d}{dt}\mu_t(f) = \mu_t\big(\mathcal{L}f + \mathcal{V}f - \mu_t(\mathcal{V})f\big) = \mu_t\big(\mathcal{L}f + \widetilde{\mathcal{L}}_{\mu_t} f\big)$$

McKean process $(\widetilde{X}(t) : t \geq 0)$ on E with generator $\mathcal{L}_\mu := \mathcal{L} + \widetilde{\mathcal{L}}_\mu$

$$\widetilde{\mathcal{L}}_\mu f(x) = \int_E \widetilde{W}(x,y)\mu(dy)\,\big(f(y) - f(x)\big)\ , \quad \mu \in \mathcal{P}(E)$$

such that $\mu\big(\widetilde{\mathcal{L}}_\mu f\big) = \mu(\mathcal{V}f) - \mu(\mathcal{V})\mu(f)$.

holds if and only if $\mu\big(\widetilde{W}(.,x) - \widetilde{W}(x,.)\big) = \mathcal{V}(x) - \mu(\mathcal{V})$

McKean interpretation

$$\frac{d}{dt}\mu_t(f) = \mu_t\big(\mathcal{L}f + \mathcal{V}f - \mu_t(\mathcal{V})f\big) = \mu_t\big(\mathcal{L}f + \widetilde{\mathcal{L}}_{\mu_t} f\big)$$

McKean process $(\widetilde{X}(t) : t \geq 0)$ on E with generator $\mathcal{L}_\mu := \mathcal{L} + \widetilde{\mathcal{L}}_\mu$

$$\widetilde{\mathcal{L}}_\mu f(x) = \int_E \widetilde{W}(x,y)\mu(dy)\,\big(f(y) - f(x)\big)\ , \quad \mu \in \mathcal{P}(E)$$

such that $\mu\big(\widetilde{\mathcal{L}}_\mu f\big) = \mu(\mathcal{V}f) - \mu(\mathcal{V})\mu(f)$.

Sufficient condition $\widetilde{W}(y,x) - \widetilde{W}(x,y) = \mathcal{V}(x) - \mathcal{V}(y)$

Examples for constant $c \in \mathbb{R}$ or $c = \mu(\mathcal{V})$ (using $g = g^+ - g^-$)

1. $\widetilde{W}_1(x,y) = \big(\mathcal{V}(x) - c\big)^- + \big(\mathcal{V}(y) - c\big)^+$
2. $\widetilde{W}_2(x,y) = \big(\mathcal{V}(y) - \mathcal{V}(x)\big)^+$
3. $\widetilde{W}_3(x,y) = \dfrac{\big(\mathcal{V}(x) - \mu(\mathcal{V})\big)^- \big(\mathcal{V}(y) - \mu(\mathcal{V})\big)^+}{\mu\big(\mathcal{V} - \mu(\mathcal{V})\big)^+}$

Particle approximations

$(\underline{\xi}_t : t \geq 0)$ on E^M with $\mathbb{P}^M$, $\mathbb{E}^M$, M 'clones' $\xi_t^1, \ldots \xi_t^M$ run in parallel

Estimate μ_t by the **empirical measure** $\mu_t^M := m(\underline{\xi}_t)$ with

$$m(\underline{x})(dy) := \frac{1}{M} \sum_{i=1}^{M} \delta_{x_i}(dy) \quad \text{as a distribution on } E \ .$$

Particle approximations

$(\underline{\xi}_t : t \geq 0)$ on E^M with $\mathbb{P}^M$, $\mathbb{E}^M$, M 'clones' $\xi_t^1, \ldots \xi_t^M$ run in parallel

Estimate μ_t by the **empirical measure** $\mu_t^M := m(\underline{\xi}_t)$ with

$$m(\underline{x})(dy) := \frac{1}{M} \sum_{i=1}^{M} \delta_{x_i}(dy) \quad \text{as a distribution on } E \text{ .}$$

Mean-field approximations for a given McKean model $\mathcal{L}_\mu = \mathcal{L} + \widetilde{\mathcal{L}}_\mu$

$$\mathcal{L}^M F(\underline{x}) = \sum_{i=1}^{M} \mathcal{L}_{m(\underline{x})}^{(i)} F(\underline{x}) \ , \quad F \in C_b(E^M)$$

where $\mathcal{L}_{m(\underline{x})}^{(i)}$ acts on $x^i \mapsto F(x^1, .., x^i, .., x^M)$.

Particle approximations

$(\underline{\xi}_t : t \geq 0)$ on E^M with $\mathbb{P}^M$, $\mathbb{E}^M$, M 'clones' $\xi_t^1, \ldots \xi_t^M$ run in parallel

Estimate μ_t by the **empirical measure** $\mu_t^M := m(\underline{\xi}_t)$ with

$$m(\underline{x})(dy) := \frac{1}{M} \sum_{i=1}^{M} \delta_{x_i}(dy) \quad \text{as a distribution on } E \ .$$

Mean-field approximations for a given McKean model $\mathcal{L}_\mu = \mathcal{L} + \widetilde{\mathcal{L}}_\mu$

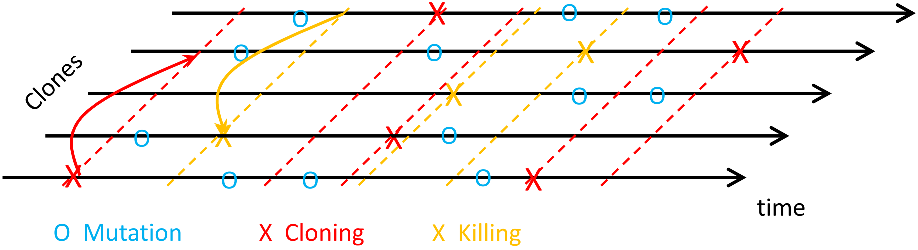
$$\mathcal{L}^M F(\underline{x}) = \sum_{i=1}^{M} \mathcal{L}^{(i)}_{m(\underline{x})} F(\underline{x}) \ , \quad F \in C_b(E^M)$$

where $\mathcal{L}^{(i)}_{m(\underline{x})}$ acts on $x^i \mapsto F(x^1, .., x^i, .., x^M)$.

$$F(\underline{x}) = f(x_i) \quad \Rightarrow \quad \mathcal{L}^M f(x_i) = \mathcal{L}_{m(\underline{x})} f(x_i)$$

$$F(\underline{x}) = \frac{1}{M} \sum_{i=1}^{M} f(x_i) = m(\underline{x})(f) \quad \Rightarrow \quad \mathcal{L}^M m(\underline{x})(f) = m(\underline{x})\big(\mathcal{L}_{m(\underline{x})} f\big)$$

Cloning/Killing interpretation



$$\mathcal{L}^M F(\underline{x}) = \sum_{i=1}^{M} \int_E W(x_i, dy)\big(F(\underline{x}^{i,y}) - F(\underline{x})\big)$$
$$+ \sum_{i=1}^{M} \big(\mathcal{V}(x_i) - c\big)^+ \frac{1}{M} \sum_{j=1}^{M} \big(F(\underline{x}^{j,x_i}) - F(\underline{x})\big)$$
$$+ \sum_{i=1}^{M} \big(\mathcal{V}(x_i) - c\big)^- \frac{1}{M} \sum_{j=1}^{M} \big(F(\underline{x}^{i,x_j}) - F(\underline{x})\big)$$

Martingale characterization

Stochastic analysis to characterize fluctuations:

$$\mathcal{M}_F^M(t) := F(\underline{\xi}_t) - F(\underline{\xi}_0) - \int_0^t \mathcal{L}^M F(\underline{\xi}_s)ds$$

is a **martingale** on $\mathbb{R}$ with (predictable) **quadratic variation**

$$\langle \mathcal{M}_F^M \rangle(t) = \int_0^t \Gamma^M F(\underline{\xi}_s)ds \ ,$$

and **carré du champ** $\Gamma^M F(\underline{x}) := \mathcal{L}^M F^2(\underline{x}) - 2F(\underline{x})\mathcal{L}^M F(\underline{x})$.

Martingale characterization

Stochastic analysis to characterize fluctuations:

$$\mathcal{M}_F^M(t) := F(\underline{\xi}_t) - F(\underline{\xi}_0) - \int_0^t \mathcal{L}^M F(\underline{\xi}_s)\, ds$$

is a **martingale** on $\mathbb{R}$ with (predictable) **quadratic variation**

$$\langle \mathcal{M}_F^M \rangle(t) = \int_0^t \Gamma^M F(\underline{\xi}_s)\, ds ,$$

and **carré du champ** $\Gamma^M F(\underline{x}) := \mathcal{L}^M F^2(\underline{x}) - 2F(\underline{x})\mathcal{L}^M F(\underline{x})$.

Mean-field approximations with McKean model $\mathcal{L}_\mu$ and Γ_μ

$$F(\underline{x}) = f(x_i) \quad \Rightarrow \quad \Gamma^M F(\underline{x}) = \Gamma_{m(\underline{x})} f(x_i)$$

$$F(\underline{x}) = m(\underline{x})(f) \quad \Rightarrow \quad \Gamma^M F(\underline{x}) = \frac{1}{M} m(\underline{x})(\Gamma_{m(\underline{x})} f) .$$

Main convergence result

Assumptions

For a given McKean model $\mathcal{L}_\mu$ on E we assume that the sequence of particle approximations $(\underline{\xi}_t : t \geq 0)$ on E^M with generators $\mathcal{L}^M$ (and associated Γ^M) satisfy for all $f \in C_b(E)$ and $F(\underline{x}) = m(\underline{x})(f)$

(A1) $\quad \mathcal{L}^M F(\underline{x}) = m(\underline{x})\big(\mathcal{L}_{m(.)} f\big)$ for all $M \geq K$

(Consistency)

(A2) $\quad \Gamma^M F(\underline{x}) = \dfrac{1}{M} m(\underline{x})\big(G_{m(.)} f\big) + O\big(\frac{1}{M^2}\big)$ as $M \to \infty$

where $\sup\limits_{\mu \in \mathcal{P}(E)} \sup\limits_{\|f\| \leq 1} \|G_\mu(f, f)\| < \infty$

(Concentration)

(A3) almost surely, $\sup\limits_{t \geq 0} \big|\{1 \leq i \leq M : \xi_t^i \neq \xi_{t-}^i\}\big| \leq K$

(bounded jumps)

(A4) $\quad \xi_0^1, \ldots, \xi_0^M \sim \mu_0$ i.i.d.r.v.s

(initial condition)

Main convergence result

Convergence of estimators (LLN)

Assume asymptotic stability and (A1) to (A4) for a particle approximation μ_t^M. Then for all $f \in C_b(E)$ the systematic error is bounded as

$$\sup_{t\geq 0} \left| \mathbb{E}^M \left[\mu_t^M(f) \right] - \mu_t(f) \right| \leq \frac{C\|f\|}{M} \quad \text{for all } M \text{ large enough },$$

where $\mu_t(f)$ solves $(\star)$, and the random error for all $p \geq 2$ as

$$\sup_{t\geq 0} \mathbb{E}^M \left[\left| \mu_t^M(f) - \mathbb{E}^M \left[\mu_t^M(f) \right] \right|^p \right]^{1/p} \leq \frac{C_p\|f\|}{\sqrt{M}} .$$

Main convergence result

Convergence of estimators (LLN)

Assume asymptotic stability and (A1) to (A4) for a particle approximation μ_t^M. Then for all $f \in C_b(E)$ the systematic error is bounded as

$$\sup_{t\geq 0} \left| \mathbb{E}^M \left[\mu_t^M(f) \right] - \mu_t(f) \right| \leq \frac{C\|f\|}{M} \quad \text{for all } M \text{ large enough },$$

where $\mu_t(f)$ solves $(\star)$, and the random error for all $p \geq 2$ as

$$\sup_{t\geq 0} \mathbb{E}^M \left[\left| \mu_t^M(f) - \mathbb{E}^M \left[\mu_t^M(f) \right] \right|^p \right]^{1/p} \leq \frac{C_p\|f\|}{\sqrt{M}} .$$

Unbiased estimators for unnormalized measures

$$\nu_t^M(f) := \nu_t^M(1)\mu_t^M(f) \quad \text{with} \quad \nu_t^M(1) := \exp\left(\int_0^t \mu_s^M(\mathcal{V})ds \right) ,$$

where $\mathbb{E}^M \left[\nu_t^M(f) \right] = \nu_t(f)$ for all $t \geq 0$, $M \geq 1$, $f \in C_B(E)$.

Sketch of the proof

- **propagator** $\Theta_{t,T} f(x) := \frac{P^{\mathcal{V}}_{T-t} f(x)}{\mu_t(P^{\mathcal{V}}_{T-t} 1)}$ s. th. $\mu_T(f) = \mu_t\big(\Theta_{t,T} f\big)$
- $$\mathbb{E}^M\left[\left|\mu_T^M(f) - \mu_T(f)\right|^p\right]^{1/p} \leq E^M\left[\left|\mu_T^M(f) - \mu_t^M\left(\Theta_{t,T} f\right)\right|^p\right]^{1/p} + E^M\left[\left|\mu_t^M\left(\Theta_{t,T} f\right) - \mu_T(f)\right|^p\right]^{1/p}$$
- choose $T - t \propto \log M$ and use **asymptotic stability**
- **martingale decomposition** for $F(\underline{x}) = m(\underline{x})(f)$

$$\mu_T^M(f) - \mu_t^M(f) = \mathcal{M}_f^M(T-t) + \int_t^T \mathcal{L}_{m(\underline{\xi}_s)}(f) ds$$

- connect **predictable QV** with QV (bounded jumps A3)
- use **BDG inequality** to bound moments of martingales (A2)

Asymptotic variance

Partial result

Assume asymptotic stability and (A1) to (A4) for a particle approximation μ_t^M, such that for some $T > 0$ and all $0 \le s \le T$, $f \in C_b(E)$

$$\mu_s^M\big(G_{\mu_s^M}(f)\big) \to \mu_s\big(G_{\mu_s}(f)\big) \qquad \text{in a strong enough sense(?)} .$$

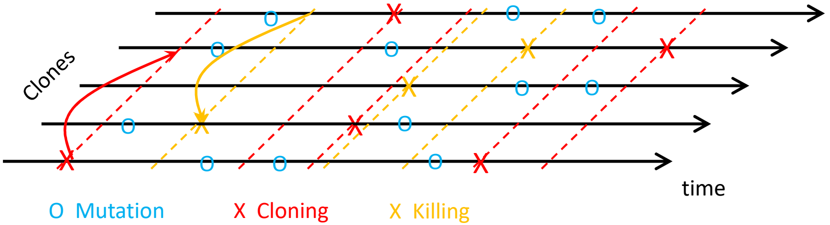
Then $V_T^M(f) := \sqrt{M}\big(\mu_T^M(f) - \mu_T(f)\big) \to V_T(f)$ in law as $M \to \infty$. Here $V_T(f)$ is a centred Gaussian with asymptotic variance

$$\mathbb{E}\big[V_T(f)^2\big] = \mu_0\big((\Theta_{0,T}\bar{f})^2\big) + \int_0^T \mu_s\big(G_{\mu_s}(\Theta_{s,T}\bar{f})\big)ds ,$$

where $\Theta_{t,T}f$ is the propagator for μ_t and $\bar{f} = f - \mu_t(f)$.

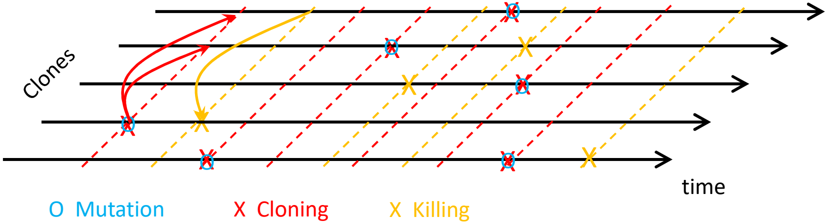
More general particle approximations

Mean-field approximation



Cloning algorithm

[Giardina, Kurchan, Peliti (2006); Lecomte, Tailleur (2007)]

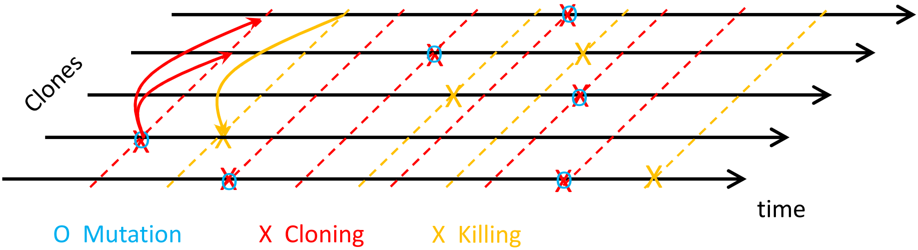


The cloning algorithm

cloning distribution π_x on subsets of $\{1, \ldots, M\}$

$$\mathcal{L}_c^M F(\underline{x}) := \sum_{i=1}^{M} \int_E W(x_i, dy) \sum_{A \subseteq \{1,..,M\}} \pi_{x_i}(A) \big(F(\underline{x}^{A,x_i;i,y}) - F(\underline{x}) \big)$$

$$+ \sum_{i=1}^{M} \big(\mathcal{V}(x_i) - c \big)^- \frac{1}{M} \sum_{j=1}^{M} \big(F(\underline{x}^{i,x_j}) - F(\underline{x}) \big)$$



The cloning algorithm

cloning distribution π_x on subsets of $\{1,\dots,M\}$

$$\mathcal{L}_c^M F(\underline{x}) := \sum_{i=1}^M \int_E W(x_i, dy) \sum_{A\subseteq\{1,..,M\}} \pi_{x_i}(A)\big(F(\underline{x}^{A,x_i;i,y}) - F(\underline{x})\big)$$

$$+\sum_{i=1}^M \big(\mathcal{V}(x_i) - c\big)^- \frac{1}{M} \sum_{j=1}^M \big(F(\underline{x}^{i,x_j}) - F(\underline{x})\big)$$

choose $\pi_x(A) = \pi_x(|A|)/\binom{M}{|A|} = 0$ if $|A| > K$ and (A3)

$$R(x) := \sum_{n=0}^M n\pi_x(n) = \frac{\big(\mathcal{V}(x) - c\big)^+}{w(x)}\ , \quad Q(x) := \sum_{n=0}^M n^2 \pi_x(n) \le C < \infty\ .$$

For $F(\underline{x}) = m(\underline{x})(f)$ we get

- $\mathcal{L}_c^M F(\underline{x}) = \mathcal{L}^M F(\underline{x}) = m(\underline{x})\big(\mathcal{L}_{m(.)} f\big)$ (A1)

The cloning algorithm

- $$\Gamma_c^M F(\underline{x}) = \frac{1}{M} m(\underline{x}) \Big(\overbrace{\Gamma_{m(.)} f + w(Q-R)\big(\ell_{m(.)} f(.)\big)^2 + O(1)}^{=G_{m(.)(f)}} \Big) + O\big(\tfrac{1}{M^2}\big)$$

  where $\ell_\mu f(x) := \int_E \big(f(y) - f(x)\big)\mu(dy)$ (A2)

The cloning algorithm

- $\Gamma_c^M F(\underline{x}) = \frac{1}{M} m(\underline{x}) \Big(\overbrace{\Gamma_{m(.)} f + w(Q-R)\big(\ell_{m(.)} f(.)\big)^2 + O(1)}^{=G_{m(.)(f)}} \Big) + O\big(\tfrac{1}{M^2}\big)$

where $\quad \ell_\mu f(x) := \int_E \big(f(y) - f(x)\big)\mu(dy)$ (A2)

generic choice $\quad \pi_x(n) = \begin{cases} R(x) - \lfloor R(x) \rfloor & , \; n = \lfloor R(x) \rfloor + 1 \\ \lfloor R(x) \rfloor + 1 - R(x) & , \; n = \lfloor R(x) \rfloor \\ 0 & , \; \text{otherwise} \end{cases}$

Selection intensities for McKean models, to minimize $\Gamma_{m(.)} f$

1. $\widetilde{W}_1(x,y) = \big(\mathcal{V}(x) - c\big)^- + \big(\mathcal{V}(y) - c\big)^+$
2. $\widetilde{W}_2(x,y) = \big(\mathcal{V}(y) - \mathcal{V}(x)\big)^+$
3. $\widetilde{W}_3(x,y) = \dfrac{\big(\mathcal{V}(x) - \mu(\mathcal{V})\big)^- \big(\mathcal{V}(y) - \mu(\mathcal{V})\big)^+}{\mu\big(\mathcal{V} - \mu(\mathcal{V})\big)^+}$

The cloning algorithm

- $\Gamma_c^M F(\underline{x}) = \frac{1}{M} m(\underline{x}) \Big(\overbrace{\Gamma_{m(.)} f + w(Q-R)\big(\ell_{m(.)} f(.)\big)^2 + O(1)}^{=G_{m(.)(f)}} \Big) + O\big(\frac{1}{M^2}\big)$

where $\quad \ell_\mu f(x) := \int_E \big(f(y) - f(x)\big)\mu(dy)$ (A2)

generic choice $\quad \pi_x(n) = \begin{cases} R(x) - \lfloor R(x) \rfloor & , \ n = \lfloor R(x) \rfloor + 1 \\ \lfloor R(x) \rfloor + 1 - R(x) & , \ n = \lfloor R(x) \rfloor \\ 0 & , \ \text{otherwise} \end{cases}$

Selection intensities for McKean models, to minimize $\Gamma_{m(.)} f$

1. $S_1(\underline{x}) = \sum_{i=1}^M \big|\mathcal{V}(x_i) - c\big| \quad \Rightarrow \quad c = \text{median}(\mathcal{V}(\underline{x}))$

2. $S_2(\underline{x}) = \frac{1}{2} \sum_{i,j=1}^M \big|\mathcal{V}(x_i) - \mathcal{V}(x_j)\big| \le S_1(\underline{x})$

3. $S_3(\underline{x}) \le S_2(\underline{x})$

Dynamic large deviations

Continuous-time Markov jump process $(X_t : t \geq 0)$ with path space $(\Omega, \mathbb{P})$

Additive path space observable , $g \in C_b(E^2)$, $h \in C_b(E)$

$$A_T[\omega] := \sum_{\substack{t \leq T \\ \omega(t-) \neq \omega(t)}} g\big(\omega(t-), \omega(t)\big) + \int_0^T h\big(\omega(t)\big) dt$$

Assume that A_T fulfills a **large deviation principle** such that

$$\mathbb{P}\big[A_T/T \approx a\big] \asymp e^{-T\, I(a)} \quad \text{as } T \to \infty ,$$

with **rate function** $I(a) \in [0, \infty]$.

Assume $I(a)$ is **convex**, then $\quad I(a) = \sup_{k \in \mathbb{R}} \big(ka - \lambda_k\big)$ for all $a \in \mathbb{R}$,

given by the Legendre transform of the **SCGF**

$$\lambda_k := \lim_{T \to \infty} \frac{1}{T} \log \mathbb{E}_{\mu_0}\big[e^{kA_T}\big] \in \mathbb{R} , \quad k \in \mathbb{R} .$$

[Chetrite, Touchette (2015)]

SCGF and tilted generator

$$\nu_T^k(f) = \mathbb{E}_{\mu_0}\Big[f(X_T)\exp\Big(k\sum_{t\le T} g\big(X_{t-},X_t\big) + k\int_0^T h\big(X_t\big)dt\Big)\Big]$$

λ_k is principal eigenvalue of the **tilted generator**

$$\mathcal{L}_k f(x) := \int_E W(x,dy)\big[e^{kg(x,y)}f(y) - f(x)\big] + kh(x)f(x)$$

$$= \underbrace{\int_E W_k(x,dy)\big[f(y)-f(x)\big]}_{\widehat{\mathcal{L}}_k f(x)} + \mathcal{V}_k(x)f(x)$$

with modified rates $W_k(x,dy) = W(x,dy)e^{kg(x,y)}$ for the jump part $\widehat{\mathcal{L}}_k$,

and diagonal potential $\mathcal{V}_k(x) = \int_E W(x,dy)\big[e^{kg(x,y)} - 1\big] + kh(x)$.

SCGF and tilted generator

$$\nu_T^k(f) = \mathbb{E}_{\mu_0}\Big[f(X_T)\exp\Big(k\sum_{t\leq T} g\big(X_{t-},X_t\big) + k\int_0^T h\big(X_t\big)dt\Big)\Big]$$

λ_k is principal eigenvalue of the **tilted generator**

$$\mathcal{L}_k f(x) := \int_E W(x,dy)\big[e^{kg(x,y)}f(y) - f(x)\big] + kh(x)f(x)$$
$$= \underbrace{\int_E W_k(x,dy)\big[f(y)-f(x)\big]}_{\widehat{\mathcal{L}}_k f(x)} + \mathcal{V}_k(x)f(x)$$

with modified rates $W_k(x,dy) = W(x,dy)e^{kg(x,y)}$ for the jump part $\widehat{\mathcal{L}}_k$,

and diagonal potential $\mathcal{V}_k(x) = \int_E W(x,dy)\big[e^{kg(x,y)} - 1\big] + kh(x)$.

- For all $\nu_0^k = \mu_0$, **asymptotic stability** implies that as $t\to\infty$

$$\lambda_k(t) := \frac{1}{t}\ln\nu_t^k(1) = \frac{1}{t}\int_0^t \mu_s^k(\mathcal{V}_k)\,ds \;\to\; \lambda_k = \mu_\infty^k(\mathcal{V}_k)$$

SCGF and tilted generator

$$\nu_T^k(f) = \mathbb{E}_{\mu_0}\Big[f(X_T)\exp\Big(k\sum_{t\le T} g\big(X_{t-},X_t\big) + k\int_0^T h\big(X_t\big)dt\Big)\Big]$$

λ_k is principal eigenvalue of the **tilted generator**

$$\mathcal{L}_k f(x) := \int_E W(x,dy)\big[e^{kg(x,y)}f(y) - f(x)\big] + kh(x)f(x)$$
$$= \underbrace{\int_E W_k(x,dy)\big[f(y)-f(x)\big]}_{\widehat{\mathcal{L}}_k f(x)} + \mathcal{V}_k(x)f(x)$$

with modified rates $W_k(x,dy) = W(x,dy)e^{kg(x,y)}$ for the jump part $\widehat{\mathcal{L}}_k$,

and diagonal potential $\mathcal{V}_k(x) = \int_E W(x,dy)\big[e^{kg(x,y)} - 1\big] + kh(x)$.

- Furthermore, with $\lambda_k(u,t) := \frac{1}{t-u}\int_u^t \mu_s^k(\mathcal{V}_k)\,ds$

$$\big|\lambda_k(bt,t) - \lambda_k\big| \le C\|\mathcal{V}_k\|\frac{\alpha^{bt}}{t} \quad \text{for all } b\in[0,1)\ .$$

Particle estimators for SCGF

particle approximation $(\underline{\xi}_t : t \geq 0)$, with empirical measure μ_t^M

Estimator $\Lambda_k^M(u,t) := \frac{1}{t-u}\int_u^t ds \frac{1}{M}\sum_{i=1}^{M} \mathcal{V}_k(\xi_s^i)$

Particle estimators for SCGF

particle approximation $(\underline{\xi}_t : t \geq 0)$, with empirical measure μ_t^M

Estimator $\Lambda_k^M(u,t) := \frac{1}{t-u} \int_u^t ds \frac{1}{M} \sum_{i=1}^{M} \mathcal{V}_k(\xi_s^i)$

Error bounds for cloning algorithm

Assume asymptotic stability and (A1) to (A4). Then there exist $C, C_p > 0$ such that for all $b \in [0,1)$ the systematic error is

$$\left|\mathbb{E}^M \left[\Lambda_k^M(bt,t)\right] - \lambda_k\right| \leq C\|\mathcal{V}_k\|\left(\frac{1}{M} + \frac{\alpha^{bt}}{t}\right) \quad \text{for all } M, t \text{ large enough} ,$$

and the random error for all $p > 2$ is

$$\sup_{t\geq 0} \mathbb{E}^M \left[\left|\Lambda_k^M(t) - \mathbb{E}^M\left[\Lambda_k^M(t)\right]\right|^p\right]^{1/p} \leq \frac{C_p\|\mathcal{V}_k\|}{\sqrt{M}} \quad \text{for all } M \text{ large enough} .$$

The cloning factor

Jump process $\big(C_t^M : t \geq 0\big)$ on $\mathbb{R}^+$ with $C_0^M = 1$, where at each selection event of size $n \geq -1$ at time t the value is updated as

$$C_t^M = C_{t-}^M (1 + n/M) \ .$$

- joint process $\big(\underline{\xi}_t, C_t^M\big)$ is Markov with **extended generator** $\overline{\mathcal{L}}_c^M F(\underline{x}, c)$
- $\mathbb{E}^M [C_t^M]\, e^{tc} = \nu_t(1)$, **unbiased estimator** for $e^{\lambda_k t}$ for large t

The cloning factor

Jump process $\left(C_t^M : t \geq 0\right)$ on $\mathbb{R}^+$ with $C_0^M = 1$, where at each selection event of size $n \geq -1$ at time t the value is updated as

$$C_t^M = C_{t-}^M (1 + n/M) \; .$$

- joint process $\left(\underline{\xi}_t, C_t^M\right)$ is Markov with **extended generator** $\overline{\mathcal{L}}_c^M F(\underline{x}, c)$
- $\mathbb{E}^M [C_t^M]\, e^{tc} = \nu_t(1)$, **unbiased estimator** for $e^{\lambda_k t}$ for large t
- Observing only the cloning factor with $F(\underline{x}, c) = \log c$ we get

$$\overline{\mathcal{L}}_c^M F(\underline{x}, c) = m(\underline{x})(\mathcal{V}_k - c) + O\Big(\frac{1}{M}\Big)$$

due to bounded clone event size n (A4)

The cloning factor

Jump process $\big(C_t^M : t \geq 0\big)$ on $\mathbb{R}^+$ with $C_0^M = 1$, where at each selection event of size $n \geq -1$ at time t the value is updated as

$$C_t^M = C_{t-}^M (1 + n/M) \ .$$

- joint process $\big(\underline{\xi}_t, C_t^M\big)$ is Markov with **extended generator** $\overline{\mathcal{L}}_c^M F(\underline{x}, c)$
- $\mathbb{E}^M[C_t^M]\, e^{tc} = \nu_t(1)$, **unbiased estimator** for $e^{\lambda_k t}$ for large t
- Observing only the cloning factor with $F(\underline{x}, c) = \log c$ we get

$$\overline{\mathcal{L}}_c^M F(\underline{x}, c) = m(\underline{x})(\mathcal{V}_k - c) + O\Big(\frac{1}{M}\Big)$$

  due to bounded clone event size n (A4)
- Martingale characterization provides **new estimator**

$$\bar{\Lambda}_k^M(u,t) := \frac{1}{t-u} \log \frac{C_t^M}{C_u^M} + c = \Lambda_k^M(u,t) + \frac{\mathcal{M}_C^M(t) - \mathcal{M}_C^M(u)}{t-u} + O\big(\tfrac{1}{M}\big)$$

Current large deviations for IPS

- IPS with generator
$$\mathcal{L}f(\eta) = \sum_{z\in\mathbb{T}_L} p\, u(\eta_z, \eta_{z+1})\big(f(\eta^{z,z+1}) - f(\eta)\big) + q\, u(\eta_z, \eta_{z-1})\big(f(\eta^{z,z-1}) - f(\eta)\big)$$
cont.-time jump process on state space $E = \mathbb{N}_0^L$, NN dynamics with PBC
- finite state CTMC with fixed number of particles
$\Rightarrow$ LDP for $(A_T)_T$ due to contraction [Bertini, Faggionato, Gabrielli (2015)]

Current large deviations for IPS

- IPS with generator $$\mathcal{L}f(\eta) = \sum_{z\in\mathbb{T}_L} p\,u(\eta_z,\eta_{z+1})\big(f(\eta^{z,z+1}) - f(\eta)\big) + q\,u(\eta_z,\eta_{z-1})\big(f(\eta^{z,z-1}) - f(\eta)\big)$$

  cont.-time jump process on state space $E = \mathbb{N}_0^L$, NN dynamics with PBC

- finite state CTMC with fixed number of particles

  $\Rightarrow$ LDP for $(A_T)_T$ due to contraction [Bertini, Faggionato, Gabrielli (2015)]

- assume $\sum_z u(\eta_z,\eta_{z+1}) = \sum_z u(\eta_z,\eta_{z-1})$ and $p+q=1$

  total exit rate $w(\eta) = \sum_{z\in\mathbb{T}_L} u(\eta_z,\eta_{z+1})$

  $w_k(\eta) = Q_k w(\eta)$ where $Q_k = pe^k + qe^{-k}$

  potential $\mathcal{V}_k(\eta) = (Q_k - 1)w(\eta)$

Current large deviations for IPS

- IPS with generator $$\mathcal{L}f(\eta) = \sum_{z\in\mathbb{T}_L} p\, u(\eta_z,\eta_{z+1})\big(f(\eta^{z,z+1}) - f(\eta)\big) + q\, u(\eta_z,\eta_{z-1})\big(f(\eta^{z,z-1}) - f(\eta)\big)$$

  cont.-time jump process on state space $E = \mathbb{N}_0^L$, NN dynamics with PBC

- finite state CTMC with fixed number of particles

  $\Rightarrow$ LDP for $(A_T)_T$ due to contraction [Bertini, Faggionato, Gabrielli (2015)]

- assume $\sum_z u(\eta_z,\eta_{z+1}) = \sum_z u(\eta_z,\eta_{z-1})$ and $p+q=1$

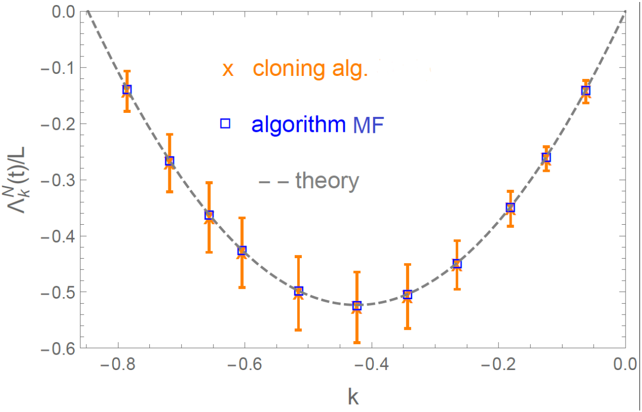
  total exit rate $w(\eta) = \sum_{z\in\mathbb{T}_L} u(\eta_z,\eta_{z+1})$

  $w_k(\eta) = Q_k w(\eta)$ where $Q_k = pe^k + qe^{-k}$

  potential $\mathcal{V}_k(\eta) = (Q_k - 1)w(\eta)$

Example inclusion process with $u(n,m) = n(d+m)$

Comparison of algorithms

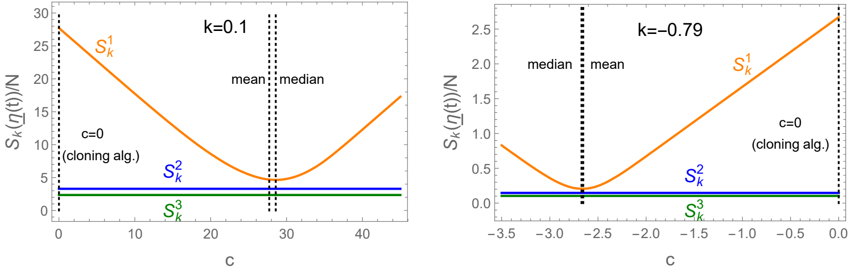


SCGF for current fluctuations in a stochastic lattice gas

cloning algorithm $\widetilde{W}(x,y) = \big(\mathcal{V}_k(y) - c\big)^+ + \big(\mathcal{V}_k(x) - c\big)^-,\ c = 0$

mean-field approximation $\widetilde{W}(x,y) = \big(\mathcal{V}(y) - \mathcal{V}(x)\big)^+$

Comparison of algorithms



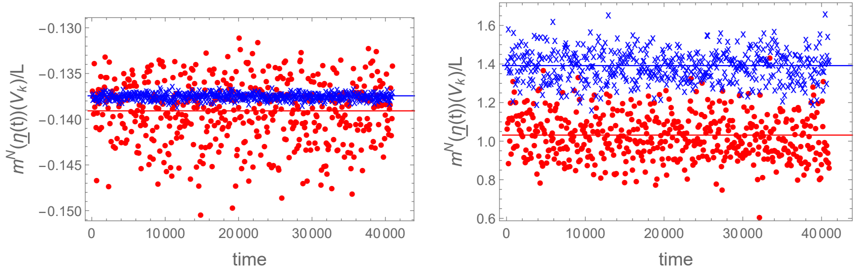
total selection intensities (depending on McKean models)

cloning algorithm $S_1^M(\eta) = |Q_k - 1| w(\eta)$

mean-field $S_2^M(\underline{x}) = |Q_k - 1| \frac{1}{2M} \sum_{i,j} \left| w(\eta^i) - w(\eta^j) \right|$

$$S_3^M(\underline{x}) = |Q_k - 1| \frac{1}{2} \sum_i \left| w(\eta^i) - \mu(\underline{\eta})(w) \right|$$

Comparison of algorithms



$k = -0.79$ $\qquad$ $k = 0.1$

time series $\mu_t^M(\mathcal{V}_k)$, $\qquad \Lambda_k^M(u,t) = \dfrac{1}{t-u}\displaystyle\int_u^t \mu_s^M(\mathcal{V}_k)ds$

cloning algorithm $\qquad$ mean-field approximation

Conclusion

Summary

- analyze and compare particle approximations via martingale characterizations
- adapt/generalize convergence results from sequential Monte Carlo
  [del Moral, Miclo, Rousset,. . .]
- rigorous version of recent heuristic results on cloning algorithms
  [Nemoto, Guevara Hidalgo, Lecomte (2017), Guevara Hidalgo (2018)]

Work in progress/open questions

- asymptotic covariances and convergence to a stationary Gaussian process
- extensions and potential improvement of cloning algorithms
  exploit freedom in McKean representations and particle approximations
- numerical estimates for asymptotic variances

Thank you!