Capital Requirements and Claims Recovery: A New Perspective on Solvency Regulation

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Motivation

1. **Protection of creditors** is a key objective of financial regulation.
2. The current regulatory requirements based on Value at Risk and Average Value at Risk limit the probability of default of financial institutions, **but fail to control the size of recovery** on creditors' claims in the case of default.
3. **We resolve this failure** by developing novel recovery risk measures.
4. The new risk measures can be applied to **performance-based management of business divisions and optimal portfolio choice**.
5. **Simulation algorithms for computing risk measure that quantify rare events** are available in the literature. These can be extended to recovery risk measures.

Outline

1 Solvency Capital Regulation
- Risk-Sensitive Solvency Regimes
- Claims Recovery

2 Recovery Risk Measures
- Introducing Recovery Risk Measures
- Controlling Recovery
- Applications

3 Conclusion

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Outline

1. **Solvency Capital Regulation**
   - Risk-Sensitive Solvency Regimes
   - Claims Recovery
2. **Recovery Risk Measures**
   - Introducing Recovery Risk Measures
   - Controlling Recovery
   - Applications
3. **Conclusion**

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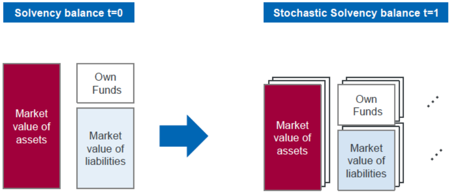
Outline

Computation of Capital Requirements

- One-period model with dates $t = 0, 1$ and market-consistent valuation of all assets and liabilities
  - Time 0: today
  - Time 1: regulatory time horizon
- The own funds, or net asset values, are denoted by E_0 and E_1.
  - The own funds E_0 at $t = 0$ are deterministic.
  - The own funds E_1 on a one-year horizon are random.
- Stochastic projections capture the random evolution of the balance sheet.

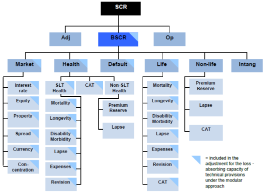


Solvency II – Standard Formula vs. Internal Model

Methods for SCR calculation

- Standard Formula
- Standard Formula with company specific parameter
- Partial Internal Model
- Internal Model

Standard Formula



- Calculation of capital requirements per sub-modules based on prescribed stress levels
- Aggregation of single risks via iterative application of the square root formula according to a fixed correlation matrix

Solvency II – Standard Formula vs. Internal Model (cont.)

Methods for SCR calculation

- Standard Formula
- Standard Formula with company specific parameter
- Partial Internal Model
- Internal Model

Internal Models

- SCR calculation with complex simulation models, partially with approximation techniques, e. g., replicating portfolios or LSMC for Internal Models Life
- Approval by supervisory authority required
- Internal Models are more than just a tool to compute the SCR:
  Embedding of the Internal Model into the corporate management necessary ('Use Test')

Solvency Tests

- **Solvency balance sheet**

Assets	Liabilities
A_t	L_t
	$E_t = A_t - L_t$

  - The quantities at time $t = 0$ are known whereas the quantities at time $t = 1$ are random variables on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
  - The increment of the net asset value is $\Delta E_1 := E_1 - E_0$.

- **Solvency test**

  For a given regulatory monetary risk measure ρ, the company is solvent[1] if

$$\rho(\Delta E_1) \leq E_0 \quad \Longleftrightarrow \quad \rho(E_1) \leq 0$$

[1] In practice, solvency capital requirements may only refer to "unexpected" losses. In this case, in the definition of ΔE_1, E_0 is replaced by the expected value of (the suitably discounted) E_1. In this respect, the European regulatory framework for insurance companies Solvency II is contradictory in itself.

Solvency Tests (cont.)

- **Standard examples** are the monetary risk measures *Value at Risk* (V@R) and *Average Value at Risk* (AV@R) at some level $\alpha \in (0,1)$:

$$\mathrm{V@R}_\alpha(X) := \inf\{x \in \mathbb{R};\ P(X + x < 0) \leq \alpha\}, \qquad \mathrm{AV@R}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \mathrm{V@R}_\beta(X)d\beta$$

- **Regulatory standards**
  - **Solvency II** V@R at level $\alpha = 0.5\%$
  - **Swiss solvency test** AV@R at level $\alpha = 1\%$
  - **Basel III** AV@R with level $\alpha = 2.5\%$
- **Simulation of risk measures**
  - Applications of stochastic approximation and stochastic average approximation to the evaluation of risk measures were investigated by Rockafellar & Uryasev (2000), Rockafellar & Uryasev (2002), Dunkel & Weber (2007), Bardou, Frikha & Pagès (2009), Dunkel & Weber (2010), Meng, Sun & Goh (2010), Sun, Xu & Wang (2014), Bardou, Frikha & Pagès (2016), Kim & Weber (2021).

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Axiomatic Theory of Risk Measures

- Artzner, Delbaen, Eber & Heath (1999)
- Föllmer & Schied (2002)
- Frittelli & Rosazza Gianin (2002)
- ...

Happy Birthday!

Hans Föllmer
(born May 20, 1941)

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Outline

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Recovery of Claims

Recovery of Claims

The event

$$\{A_1 \geq \lambda L_1\}$$

contains those scenarios for which at least a fraction $\lambda \in (0,1)$ of the claims is recovered.

Question

- Which risk measures control the probability $P(A_1 \geq \lambda L_1)$ of recovering at least a fraction $\lambda \in (0,1)$ of claims?
  - Both V@R and AV@R fail at this task.
  - We suggest recovery risk measures that control recovery and can serve as a basis for solvency tests.
  - They can also be applied to performance-based management of business divisions of firms and optimal investment management.

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Recovery of Claims (2)

Proposition

We denote by $\mathcal{X}$ a set of positive random variables on some nonatomic probability space $(\Omega, \mathcal{F}, P)$. We assume that $\mathcal{X}$ contains all positive discrete random variables.

For all $\alpha \in (0,1)$ and $\lambda \in (0,1)$ we have

$$\begin{aligned} 1-\alpha &= \inf\{P(A \geq \lambda L)\,;\; A, L \in \mathcal{X},\; \mathrm{AV@R}_\alpha(A-L) \leq 0\} \\ &= \inf\{P(A \geq \lambda L)\,;\; A, L \in \mathcal{X},\; \mathrm{V@R}_\alpha(A-L) \leq 0\}. \end{aligned}$$

Recovery of Claims – Example

- Probability space $\Omega = \{g, b\}$ with $\mathbb{P}(b) = \frac{\alpha}{2}$ with $\alpha \approx 0$, say $\alpha = 0.5\%$ or $\alpha = 1\%$.
- Liabilities

$$L_1(\omega) = \begin{cases} 1 & \text{if } \omega = g, \\ 100 & \text{if } \omega = b. \end{cases}$$

- The company can manage its assets by engaging in a stylized financial contract with zero initial cost transferring dollars from the good state to the bad state.
- More specifically, we assume that the company can choose one of the following asset profiles at time 1:

$$A_1^k(\omega) = \begin{cases} 101 - k & \text{if } \omega = g, \\ k & \text{if } \omega = b, \end{cases} \quad \text{with} \quad k \in [0, 100].$$

- Hedging its liabilities completely would require the company to choose $k = 100$.

Recovery of Claims – Example (2)

- For any $k \in [0, 100]$, the company's net asset value is given by

$$E_1^k(\omega) = \begin{cases} 100 - k & \text{if } \omega = g, \\ k - 100 & \text{if } \omega = b. \end{cases}$$

- Due to limited liability, the corresponding shareholder value is

$$\max\{E_1^k(\omega), 0\} = \begin{cases} 100 - k & \text{if } \omega = g, \\ 0 & \text{if } \omega = b. \end{cases}$$

- Hence, the choice $k = 0$ is optimal from the perspective of shareholders – corresponding to no recovery in the bad state.
- The company is solvent independent of k:

$$\mathrm{V@R}_\alpha(E_1^k) = k - 100 \leq 0, \quad \mathrm{AV@R}_\alpha(E_1^k) = \frac{1}{\alpha}\left(\frac{\alpha}{2}(100 - k) + \frac{\alpha}{2}(k - 100)\right) = 0.$$

Outline

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Outline

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Recovery Value at Risk

Definition

The **Recovery Value at Risk** with increasing *level function* $\gamma : [0,1] \to (0,1)$ is defined by

$$\mathrm{RecV@R}_\gamma(X, Y) = \sup_{\lambda \in [0,1]} \mathsf{V@R}_{\gamma(\lambda)}(X + (1-\lambda)Y).$$

- Consider solvency balance sheets, for times $t = 0, 1$,

Assets	Liabilities
A_t	L_t
	$E_t = A_t - L_t$

  and set $\Delta E_1 = E_1 - E_0$.

- The **solvency condition** $\mathrm{RecV@R}_\gamma(\Delta E_1, L_1) \leq E_0$ is **equivalent** to requiring that for **all recovery fractions** $\lambda \in [0,1]$ the recovery probabilities satisfy

$$P(A_1 < \lambda L_1) \leq \gamma(\lambda) \iff P(A_1 \geq \lambda L_1) \geq 1 - \gamma(\lambda).$$

Recovery Value at Risk (cont.)

Selected properties

1. **Cash invariance in the first component**

$$\mathrm{RecV@R}_\gamma(X+m, Y) = \mathrm{RecV@R}_\gamma(X, Y) - m \qquad \forall\, X, Y \in L^0,\ m \in \mathbb{R}$$

2. **Monotonicity**

$$X_1 \geq X_2,\ \ Y_1 \geq Y_2 \quad \Longrightarrow \quad \mathrm{RecV@R}_\gamma(X_1, Y_1) \leq \mathrm{RecV@R}_\gamma(X_2, Y_2) \qquad \forall\, X_1, X_2, Y_1, Y_2 \in L^0$$

3. **Positive homogeneity**

$$\mathrm{RecV@R}_\gamma(aX, aY) = a \cdot \mathrm{RecV@R}_\gamma(X, Y) \qquad \forall\, X, Y \in L^0,\ a \in [0, \infty)$$

4. **Star-shapedness in the first component**

$$\mathrm{RecV@R}_\gamma(aX, Y) \geq a \cdot \mathrm{RecV@R}_\gamma(X, Y) \qquad \forall\, X, Y \in L^0,\ Y \geq 0,\ a \in [1, \infty)$$

General Recovery Risk Measures

Definition

For every $\lambda \in [0,1]$ consider a map $\rho_\lambda : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ and assume that $\rho_{\lambda_1} \geq \rho_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$. The **recovery risk measure** $\mathrm{Rec}\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is defined by $\mathrm{Rec}\rho(X, Y) := \sup_{\lambda \in [0,1]} \rho_\lambda(X + (1-\lambda)Y)$.

- This definition contains many recovery risk measures that may possess additional desirable properties.
- An example is **recovery average value at risk** which is cash invariant in its first component, monotone, convex, subadditive, positively homogeneous, star shaped in its first component, and normalized.

Definition

The **Recovery Average Value at Risk** $\mathrm{RecAV@R}_\gamma : L^1 \times L^1 \to \mathbb{R} \cup \{\infty\}$ with increasing level function $\gamma : [0,1] \to (0,1)$ is defined by

$$\mathrm{RecAV@R}_\gamma(X, Y) := \sup_{\lambda \in [0,1]} \mathrm{AV@R}_{\gamma(\lambda)}(X + (1-\lambda)Y).$$

Subadditivity and Limit Systems

- We consider a bank or an insurance company that consists of N subentities. For each date $t = 0, 1$ their assets, liabilities, and net asset value are denoted by A_t^i, L_t^i, and E_t^i, $i = 1, \ldots, N$.
- The consolidated figures are denoted by

$$A_t = \sum_{i=1}^{N} A_t^i, \quad L_t = \sum_{i=1}^{N} L_t^i, \quad E_t = \sum_{i=1}^{N} E_t^i.$$

- The firm may enforce entity-based risk constraints of the form

$$\mathrm{RecAV@R}_\gamma(E_1^i, L_1^i) \leq c^i, \quad i = 1, \ldots, N,$$

  where $c^1, \ldots, c^N \in \mathbb{R}$ are given risk limits.
- If the limits are chosen to satisfy $\sum_{i=1}^{N} c^i \leq 0$, then by subadditivity:

$$\mathrm{RecAV@R}_\gamma(E_1, L_1) \leq \sum_{i=1}^{N} \mathrm{RecAV@R}_\gamma(E_1^i, L_1^i) \leq \sum_{i=1}^{N} c^i \leq 0.$$

Outline

Recovery of Claims – Example (cont.)

- We return to the simple example introduced above with

$$E_1^k(\omega) = \begin{cases} 100 - k & \text{if } \omega = g, \\ k - 100 & \text{if } \omega = b. \end{cases} \qquad \forall\, k \in [0, 100]$$

- For a probability level $\beta \in (0, \alpha/2)$ and a desired recovery level $r \in (0, 1)$, we set

$$\gamma(\lambda) = \begin{cases} \beta & \text{if } \lambda \in [0, r), \\ \alpha & \text{if } \lambda \in [r, 1]. \end{cases}$$

- This implies that

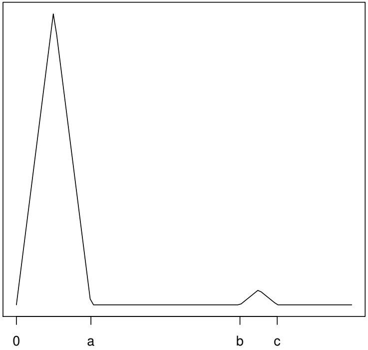
$$\mathrm{RecV@R}_\gamma(E_1^k, L_1) \leq 0 \iff k \geq 100r,$$

  i.e., the maximal shareholder value is attained for $k = 100r$.

- Liabilities in state b are equal to 100. This implies that the recovery fraction in state b is equal to r.

ALM and Recovery – Another Example

- We consider a firm with a stylized balance sheet.
- Assets are deterministic, but the firm is capable of controlling the shape of the liability distribution:
  - A_1 is equal to a constant $k > 0$



  - Liabilities L_1 follow a probability density function with two peaks. The area below the left peak equals 99.5% while the area below the right peak equals 0.5%.

ALM and Recovery – Another Example (2)

- **Regulatory capital requirements**
  - We focus on VaR at level 0.5% and AVaR at level 1%.
  - The corresponding solvency capital requirements admit analytic solutions:

$$\rho_{reg}(\Delta E_1) = \begin{cases} V@R_{0.5\%}(\Delta E_1) = a - k + E_0, \\ AV@R_{1\%}(\Delta E_1) = \left(\frac{1}{2} - \frac{1}{3}\sqrt{\frac{\alpha}{2(1-\alpha)}}\right)a + \frac{b+c}{4} - k + E_0. \end{cases}$$

- **Recovery-based capital requirements**
  - Fixing a regulatory level $\alpha \in (0,1)$, we consider a piecewise constant recovery function

$$\gamma(\lambda) = \begin{cases} \beta & \text{if } \lambda \in [0,r) \\ \alpha & \text{if } \lambda \in [r,1] \end{cases}, \ \beta \in (0,\alpha), \ r \in (0,1), \ \alpha = 0.5\%$$

  - The solvency capital requirement corresponding to $\mathrm{RecV@R}$ is

$$\mathrm{RecV@R}_\gamma(\Delta E_1, L_1) = \max\left\{a, r\frac{b+c}{2}\right\} - k + E_0.$$

ALM and Recovery – Another Example (3)

- **Recovery adjustments**
  - In order to capture the extent to which the regulatory solvency capital requirements fail to control the recovery on liabilities we define the *recovery adjustment*

$$\mathrm{RecAdj}_\gamma(\Delta E_1, L_1) := \max\left\{\frac{\mathrm{RecV@R}_\gamma(\Delta E_1, L_1)}{\rho_{reg}(\Delta E_1)}, 1\right\}.$$

  - This quantity is the maximum of 1 and the multiplicative factor **by which regulatory requirements would have to be adjusted to guarantee the considered recovery levels**.
  - The case study will show that recovery adjustments can be very high – providing alternative evidence that conventional solvency requirements do not properly protect recovery.

ALM and Recovery – Another Example (4)

- **Recovery adjustments (cont.)**
  - We ask how large the recovery adjustments may become, if a firm's asset-liability-management is constrained by the following conditions:

(1)	Solvent profile under ρ_{reg}	$\rho_{reg}(E_1) \leq 0$
(2)	Capital requirement under ρ_{reg}	$\rho_{reg}(\Delta E_1) > 0$
(3)	Solvent profile under $\mathrm{RecV@R}_\gamma$	$\mathrm{RecV@R}_\gamma(E_1, L_1) \leq 0$
(4)	Capital requirement under $\mathrm{RecV@R}_\gamma$	$\mathrm{RecV@R}_\gamma(\Delta E_1, L_1) > 0$
(5)	$\mathrm{V@R}_\alpha$ insufficient to control claims recovery	$\mathrm{RecV@R}_\gamma(E_1, L_1) > \mathrm{V@R}_\alpha(E_1)$
(6)	Range of admissible regulatory solvency ratios	$s_{min} \leq \frac{E_0}{\rho_{reg}(\Delta E_1)} \leq s_{max}$

  - More precisely, we focus on the optimization problem

$$\max \ \mathrm{RecAdj}_\gamma(\Delta E_1, L_1) \quad \text{over } A_1 \text{ and } L_1 \text{ as specified above}$$

under the constraints (1) to (6).

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ALM and Recovery – Another Example (5)

Answer

- The optimal value of the problem is bounded from above by s_{max}.
- If $\rho_{reg} = V@R_{0.5\%}$, this upper bound is attained for every choice of γ.
- If $\rho_{reg} = AV@R_{1\%}$, this upper bound is attained for special choices of γ, e.g., when $\beta \geq \frac{\alpha}{2}$ and $\frac{1}{4}\sqrt{\frac{2\alpha}{\alpha-\beta}} < r \leq \frac{1}{4}\frac{\sqrt{2\alpha}}{\sqrt{2\alpha}-\sqrt{\alpha-\beta}}\frac{1}{\frac{1}{2}+\frac{1}{3}\sqrt{\frac{\alpha}{2(1-\alpha)}}}$.

- If companies are capable and free to manage their balance sheet, classical solvency tests cannot control recovery.
- In fact, classical solvency tests permit firms to adopt strategies that correspond to the most expensive recovery adjustments under the constraints (1) - (6).

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Outline

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Performance Measurement

- **Return on risk-adjusted capital (RoRaC)**
  - For a subadditive and positively homogeneous recovery risk measure $\mathrm{Rec}\rho$, the associated RoRaC is defined by

$$\mathrm{RoRaC}(\Delta E_1, L_1) = \frac{\mathbb{E}(\Delta E_1)}{\mathrm{Rec}\rho(\Delta E_1, L_1)}.$$

  - This quantity measures the expected return per unit of economic capital expressed in terms of the risk measure $\mathrm{Rec}\rho$.
  - The **goal of the firm** is to improve its RoRaC.
- **Subentities**
  - Denote the net asset values and liabilities of the subentities at time $t = 0, 1$ by E_t^i and L_t^i for $i = 1, \ldots, N$, respectively.
  - Total net asset values and liabilities aggregate the subentities:

$$L_t = \sum_{i=1}^{N} L_t^i, \quad E_t = \sum_{i=1}^{N} E_t^i, \quad t = 0, 1$$

Performance Measurement (cont.)

- **Suitable allocation**
  - We seek an allocation of economic capital $\kappa^i = \mathrm{Rec}\rho^{\Delta E_1, L_1}(\Delta E_1^i, L_1^i)$, $i = 1, \ldots, N$, that satisfies:
    1. **Full allocation** $\sum_{i=1}^N \kappa^i = \mathrm{Rec}\rho(\Delta E_1, L_1)$;
    2. **Diversification** $\kappa^i \leq \mathrm{Rec}\rho(\Delta E_1^i, L_1^i)$ for all $i = 1, \ldots, N$;
    3. **RoRaC-compatibility** If for some $i = 1, \ldots, N$,

$$\mathrm{RoRaC}^i := \frac{\mathbb{E}(\Delta E_1^i)}{\kappa^i} > \mathrm{RoRaC}(\Delta E_1, L_1) \quad (\text{resp. } <),$$

then there exists $\varepsilon > 0$ such that for every $h \in (0, \varepsilon)$

$$\mathrm{RoRaC}(\Delta E_1 + h\Delta E_1^i, L_1 + hL_1^i) > \mathrm{RoRaC}(\Delta E_1, L_1) \quad (\text{resp. } <).$$

- **Solution**
  - Suitable allocations are Euler allocations: $\kappa^i = \frac{d}{dh}\mathrm{Rec}\rho(\Delta E_1 + h\Delta E_1^i, L_1 + hL_1^i)_{|h=0}$
  - For suitable recovery functions they can be computed for $\mathrm{RecAV@R}$.

Portfolio Optimization

- Risk measures are an important instrument to limit downside risk in portfolio optimization problems.
  - This idea is related to the classical Markowitz problem in which standard deviation quantifies the risk. Efficient frontiers characterize the best tradeoffs between return and risk.
- Recovery risk measures may successfully be applied to portfolio optimization in practice.
  - For $\mathrm{RecAV@R}$ with suitable recovery functions the characterization of the efficient frontier may, on the basis of a suitable minimax theorem, be reduced to the minimization of a linear function on a convex polyhedron.

Portfolio Optimization (2)

- **Assets and liabilities**
  - We consider $k = 1, \ldots, K$ assets with prices S_t^k, $t = 0, 1$.
  - Random one-period returns are denoted by R^k such that

$$S_1^k = S_0^k \cdot (1 + R^k).$$

  - An investor invests a fraction $x^k > 0$ of her total budget $b > 0$ into each product k. The total value of her assets at time $t = 1$ is thus equal to

$$b \cdot \sum_{k=1}^{K} x^k (1 + R^k) \; = \; b \cdot \left(1 + \sum_{k=1}^{K} x^k R^k \right).$$

    We set $\boldsymbol{R} = (R^1, \ldots, R^K)^\top$ and $\boldsymbol{x} = (x^1, \ldots, x^K)^\top$.
  - In addition, we suppose that the investor's liabilities at time $t = 1$ amount to a random fraction Z of the initial budget, i.e., the liabilities are equal to $b \cdot Z$.

Portfolio Optimization (3)

- **Efficient frontier**
  - We are interested in optimal combinations of return and downside risk – the *efficient frontier* – but with risk measured by $\mathrm{RecAV@R}$.
  - This problem can equivalently be stated as the minimization of risk for a given expected return.
- **Expected return**
  - The expected future net asset value of the investor equals

$$b \cdot \left(1 + \sum_{k=1}^{K} x^k \, \mathbb{E}(R^k) - \mathbb{E}(Z) \right)$$

  - A target expected return can be achieved by requiring for some $c \in \mathbb{R}$ that

$$\sum_{k=1}^{K} x^k \, \mathbb{E}(R^k) = c$$

Portfolio Optimization (4)

- **Downside risk**
  - We are interested in computing and optimizing

$$\mathrm{RecAV@R}_\gamma\left(b\cdot\left[1+\sum_{k=1}^{K}x^kR^k-Z\right],bZ\right)=-b+b\cdot\mathrm{RecAV@R}_\gamma\left(\sum_{k=1}^{K}x^kR^k-Z,Z\right).$$

  - We focus on the special case of piecewise-constant recovery function. In this case, $\mathrm{RecAV@R}$ is a maximum of finitely many terms:

$$\mathrm{RecAV@R}_\gamma\left(\sum_{k=1}^{K}x^kR^k-Z,Z\right)\quad=\quad\max_{i=1,\dots,n+1}\mathsf{AV@R}_{\alpha_i}\left(\sum_{k=1}^{K}x^kR^k-r_iZ\right).$$

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Portfolio Optimization (5)

A minimax theorem

- For convenience, for $i = 1, \ldots, n+1$ we define auxiliary functions

$$\Psi^i(\boldsymbol{x}, v) = \frac{1}{\alpha_i} \cdot E\left(\left[v - \sum_{k=1}^{K} x^k R^k - r_i Z\right]^+\right) - v.$$

- This allows us to write

$$\mathrm{RecAV@R}_\gamma\left(\sum_{k=1}^{K} x^k R^k - Z, Z\right) \quad = \quad \max_{i=1,\ldots,n+1} \min_{v \in \mathbb{R}} \Psi^i(\boldsymbol{x}, v).$$

Minimax theorem

The following minimax equality holds:

$$\max_{i=1,\ldots,n+1} \min_{v \in \mathbb{R}} \Psi^i(\boldsymbol{x}, v) = \min_{v \in \mathbb{R}} \max_{i=1,\ldots,n+1} \Psi^i(\boldsymbol{x}, v).$$

Portfolio Optimization (6)

- **Computing the efficient frontier**
  - Characterizing the efficient frontier — consisting of pairs of returns and downside risk — is equivalent to minimizing the function in minimax theorem additionally over $\boldsymbol{x} \in \mathcal{X}$ where

$$\mathcal{X} = \{\boldsymbol{x} \in [0,\infty)^K;\ \sum_{k=1}^{K} x^k = 1,\ \sum_{k=1}^{K} x^k\,\mathbb{E}(R^k) = c\}$$

    is a convex polyhedron.
  - This problem can be reformulated as

$$\min_{(\boldsymbol{x},v,\Upsilon)\in\mathcal{X}\times\mathbb{R}\times\mathbb{R}} \left\{\Upsilon;\ \Psi^i(\boldsymbol{x},v) \leq \Upsilon,\ i = 1,\ldots,n+1\right\}.$$

Portfolio Optimization (7)

- **Computing the efficient frontier (cont.)**
  - In typical applications in practice, these expectations are approximated via Monte Carlo simulations.
  - This allows to reformulate the problem as a linear program:

$$
\begin{aligned}
\min\ & \Upsilon \\
\text{s.t.}\quad & \frac{1}{M \cdot \alpha_i} \cdot \sum_{m=1}^{M} u^{i,m} - v \leq \Upsilon, \quad i = 1, \ldots, n+1, \\
& u^{i,m} \geq v - \sum_{k=1}^{K} x^k R^{k,m} - r_i Z^m, \quad i = 1, \ldots, n+1,\ m = 1, \ldots, M, \\
& u^{i,m} \geq 0, \quad i = 1, \ldots, n+1,\ m = 1, \ldots, M, \\
\text{over}\quad & (\boldsymbol{x}, v, \Upsilon) \in \mathcal{X} \times \mathbb{R} \times \mathbb{R},
\end{aligned}
$$

where $(\boldsymbol{R}^1, Z^1), \ldots, (\boldsymbol{R}^M, Z^M)$ are M independent simulations of the pair $(\boldsymbol{R}, Z)$.

Outline

Conclusion

1. Recovery risk measures successfully control recovery.
2. They can successfully be applied to solvency regulation, performance-based management, and portfolio optimization.
3. Future research needs to study their implementation and simulation in complex ALM-models.

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Thank you for your attention!

1. Cosimo Munari, Stefan Weber, & Lutz Wilhelmy (2021):
   'Capital Requirements and Claims Recovery: A New Perspective on Solvency Regulation'
2. Sojung Kim & Stefan Weber (2021):
   'Simulation Methods for Robust Risk Assessment and the Distorted Mix Approach'

The papers are available at: https://www.insurance.uni-hannover.de/weber